A new inequality for the Hermite constants

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Abstract: We describe continuous increasing functions $C_n(x)$ such that $\gamma_n \geq C_n(\gamma_{n-1})$ where γ_m is Hermite's constant in dimension m. This inequality yields a new proof of the Minkowski-Hlawka bound $\Delta_n \geq \zeta(n)2^{1-n}$ for the maximal density Δ_n of $n-$dimensional lattice-packings. [1]

1 Introduction and main results

We denote by $\min(\Lambda) = \min_{\lambda\in\Lambda\setminus\{0\}}\langle\lambda,\lambda\rangle$ the minimum (defined as the squared Euclidean length of a shortest non-zero element) of an $n-$dimensional lattice $\Lambda \subset \mathbf{E}^n$ in the Euclidean vector-space $\mathbf{E}^n$ and define the *density* of Λ by

$$\Delta(\Lambda) = \sqrt{\frac{(\min\,\Lambda)^n}{4^n\,\det\,\Lambda}}V_n$$

where $V_n = \frac{\pi^{n/2}}{(n/2)!}$ stands through the whole paper for the volume of the $n-$dimensional unit-ball in $\mathbf{E}^n$. The density $\Delta(\Lambda)$ is the proportion of volume occupied by a maximal open Euclidean ball embedded in the flat torus $\mathbf{E}^n/\Lambda$ with volume $\sqrt{\det(\Lambda)}$ and having a shortest closed geodesic of length $\sqrt{\min(\Lambda)}$. The largest density $\Delta_n = \Delta(\Lambda_n)$ achieved by an $n-$dimensional lattice Λ_n is called the *maximal density* in dimension n. Related constants are the *maximal center density* $\delta_n = \Delta_n/V_n$ and the *Hermite constant* $\gamma_n = 4\delta_n^{2/n}$ in dimension n. The sequence $\gamma_1, \gamma_2, \dots$ of Hermite constants satisfies for $n \geq 3$ *Mordell's inequality*

$$\gamma_n \leq \gamma_{n-1}^{(n-1)/(n-2)}$$

which yields an upper bound for γ_n (if $n \geq 3$) in terms of γ_{n-1}. Our main result is a complementary inequality bounding γ_n from below in terms of γ_{n-1}. For the convenience of the reader we state it in three equivalent ways, either in terms of densities Δ_m, center-densities δ_m or Hermite constants γ_m in dimension m. It involves the *Möbius function* $\mu : \mathbf{N}_{>0} \longrightarrow \mathbf{Z}$ defined by $\mu(l) = (-1)^a$ for a natural integer $l \in \mathbf{N}$ which is a product of a distinct primes and by $\mu(l) = 0$ if l is divisible by the square of a prime number.

*Support from the Swiss National Science Foundation is gratefully acknowledged.

[1] Math. class.: 10E05, 10E20. Keywords: Lattice packing, Hermite constant, inequality

Theorem 1.1 *(i) The maximal densities Δ_{n-1} and Δ_n of lattice-packings in dimensions $n-1$ and $n \geq 2$ satisfy the inequality*

$$2^{n-1}\Delta_{n-1} \sum_{k=1}^{\lfloor 2\Delta_n V_{n-1}/(\Delta_{n-1}V_n)\rfloor} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right) \left(1 - \left(\frac{k\Delta_{n-1}V_n}{2\Delta_n V_{n-1}}\right)^2\right)^{(n-1)/2} \geq 1$$

where the sum $\sum_{l|k}$ is over all positive integral divisors $l \in \mathbf{N}$ of the natural integer k.

(ii) The maximal center densities δ_{n-1} and δ_n of lattice-packings in dimensions $n-1$ and $n \geq 2$ satisfy the inequality

$$2^{n-1}\delta_{n-1}\frac{\pi^{(n-1)/2}}{((n-1)/2)!} \sum_{k=1}^{\lfloor 2\delta_n/\delta_{n-1}\rfloor} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right) \left(1 - \left(\frac{k\delta_{n-1}}{2\delta_n}\right)^2\right)^{(n-1)/2} \geq 1.$$

(iii) The Hermite constants γ_{n-1} and γ_n in dimensions $n-1$ and $n \geq 2$ satisfy the inequality

$$\frac{\pi^{(n-1)/2}}{((n-1)/2)!} \sum_{k=1}^{\lfloor \gamma_n^{n/2}/\gamma_{n-1}^{(n-1)/2}\rfloor} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right) \left(\gamma_{n-1} - k^2\left(\frac{\gamma_{n-1}}{\gamma_n}\right)^n\right)^{(n-1)/2} \geq 1.$$

Consider the function

$$F_n(x,y) = \sum_{k=1}^{\lfloor \sqrt{x}\ y\rfloor} \sum_{l|k} \frac{\mu(l)}{l^{n-1}} \left(x - \left(\frac{k}{y}\right)^2\right)^{(n-1)/2},$$

defined for $x, y > 0$. It is increasing in y, strictly increasing in y for $y > \frac{1}{\sqrt{x}}$, continuous and extends continuously to $y = 0$ since $F_n(x,y) = 0$ for $y \leq \frac{1}{\sqrt{x}}$. There exists thus a continous function $x \longmapsto Y_n(x)$ such that

$$F_n(x, Y_n(x)) = \frac{((n-1)/2)!}{\pi^{(n-1)/2}}$$

for all $x > 0$. Set

$$C_n(x) = \sup_{0<\xi\leq x}(\xi\ (Y_n(\xi))^{2/n}).$$

Theorem 1.2 *Let $\tilde{\gamma}_{n-1} \leq \gamma_{n-1}$ be a lower bound for Hermite's constant in dimension $n-1$. Then*

$$\gamma_n \geq C_n(\tilde{\gamma}_{n-1}) .$$

Analogous results hold of course for $\Delta_n = 2^{-n}\gamma_n^{n/2}V_n$ and $\delta_n = 2^{-n}\gamma^{n/2}$ related to Hermite's constant $\gamma_n = 4(\Delta_n/V_n)^{2/n} = 4\delta_n^{2/n}$.

Remark 1.3 *(i) The inequality of Theorem 1.2 is tight for* $n = 2$. *For* $n = 3$, *we get from* $\delta_2 = 1/2\sqrt{3}$ *the lower bound* $0.1695 \leq \delta_3 = 1/4\sqrt{2} \sim 0.1768$. *For* $n = 9$, *the known value* $\delta_8 = 1/16$ *gives the lower bound* $\delta_9 \geq 0.0388$ *(a lattice with center-density* 0.0442 *is known), for* $n = 25$ *the known value* $\delta_{24} = 1$ *coming from the Leech lattice (see Cohn-Kumar, [3] and [4]) yields* $\delta_{25} \geq 0.657$ *(a lattice with center-density* 0.707 *is known).*

(ii) The above examples show that our inequality is better than the trivial inequality

$$\delta_n \geq \sqrt{\frac{\mu_{n-1}^n}{4^n \det(\Lambda_{n-1} \oplus \mu_{n-1}\mathbf{Z})}} = \frac{\delta_{n-1}}{2}$$

obtained by considering the orthogonal sum $\Lambda_{n-1} \oplus \mu_{n-1}\mathbf{Z}$ *of a densest* $(n-1)-$*dimensional lattice* Λ_{n-1} *with minimum* $\mu_{n-1} = min_{\lambda \in \Lambda_{n-1} \setminus \{0\}} \langle \lambda, \lambda \rangle$.

(iii) The factor

$$\sum_{l|k} \frac{\mu(l)}{l^{n-1}} = \prod_{p\ prime, p|k} \left(1 - \frac{1}{p^{n-1}}\right)$$

yields only a minor improvement for huge n *and is the analogue of a standard trick leading to the factor* $\zeta(n)$ *in the Minkowski-Hlawka bound* $\Delta_n \geq \frac{\zeta(n)}{2^{n-1}}$.

(iv) The application $x \longmapsto C_n(x)$ *of Theorem 1.2 is strictly increasing for* x *huge enough. The computations of Section 6 show in fact that one has* $C_n(x) = x(Y_n(x))^{2/n}$ *(at least for huge* n*) for all values of* x *which are of interest.*

Remark 1.4 *Starting with the inequality of assertion (ii) in Theorem 1.1, and using* $\sum_{l|k} \frac{\mu(l)}{l^{n-1}} = \prod_{p|k} \left(1 - \frac{1}{p^{n-1}}\right) < 1$ *(where the product is over all prime divisors of* k*), A. Marin pointed out the easy inequalities*

$$\begin{aligned} 1 &\leq 2^{n-1}\delta_{n-1}\frac{\pi^{(n-1)/2}}{((n-1)/2)!} \sum_{k=1}^{\lfloor 2\delta_n/\delta_{n-1} \rfloor} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right) \left(1 - \left(\frac{k\delta_{n-1}}{2\delta_n}\right)^2\right)^{(n-1)/2} \\ &\leq 2^n \delta_n V_{n-1} \sum_{k=1}^{\lfloor 2\delta_n/\delta_{n-1} \rfloor} \left(1 - \left(\frac{k\delta_{n-1}}{2\delta_n}\right)^2\right)^{(n-1)/2} \frac{\delta_{n-1}}{2\delta_n} \\ &\leq 2^n \delta_n V_{n-1} \int_0^1 \left(1 - x^2\right)^{(n-1)/2} dx = 2^n \delta_n \frac{V_n}{2} = 2^{n-1}\Delta_n \end{aligned}$$

which show $\Delta_n \geq \frac{1}{2^{n-1}}$. *This is, up to the factor* $\zeta(n)$*, the Minkowski-Hlawka bound for* Δ_n.

The following, technically more involved result yields a slightly better lower bound:

Theorem 1.5 *For all $\epsilon > 0$, there exists N such that*

$$\Delta_n \geq \frac{1-\epsilon}{2^n \ \sum_{k=1}^{\infty} e^{-k^2\pi}} \sim (1-\epsilon)\ 23.1388\ 2^{-n}$$

for all $n \geq N$.

Remark 1.6 *Theorem 1.5 is slightly better that the Minkowski-Hlawka bound which shows the existence of lattices with density at least $\zeta(n)\ 2^{1-n}$, cf. formula (14) in [5], Chapter 1. The best known bound for densities achieved by lattice packings (together with a very nice proof) seems to be due to Keith Ball and asserts the existence of $n-$dimensional lattices with density at least $2(n-1)2^{-n}\zeta(n)$, see [2]. Previously, similar results where proven by Rogers and Davenport-Rogers, see [10] and [6]. Somewhat related are also [8] and [7] which describe elementary constructions of dense sphere and lattice packings.*

The paper is organized as follows:

Section 2 introduces notations and summarizes for the convenience of the reader a few well-known facts on lattices. It contains also an easy (and seemingly not very well-known) result on integral sublattices which are orthogonal to a non-zero integral vector in $\mathbf{Z}^{n+1}$.

In Section 3 we define of $\mu-$sequences which are the main tool of this paper. Theorem 3.3 of this Section gives a quantitative (and somewhat technical) statement for extending a suitable finite $\mu-$sequence $(s_0, \ldots, s_{n-1})$ to a $\mu-$sequence $(s_0, \ldots, s_{n-1}, s_n)$. The associated $n-$dimensional lattice $(s_0, \ldots, s_n)^\perp \cap \mathbf{Z}^{n+1}$ is obtained by a close analogue of lamination with respect to its $(n-1)-$dimensional sublattice $(s_0, \ldots, s_{n-1})^\perp \cap \mathbf{Z}^n$.

Theorem 3.3 is the central result of this paper since it implies easily Theorem 1.1 and Theorem 1.2 as shown at the end of Section 3. The proof of Theorem 1.5 is more technical and given in Section 6.

Section 4 states and proves a weaker and easier statement than Theorem 3.3. Although not necessary for the other parts of the paper, this section describes a fairly elementary and almost effective method for constructing dense lattices. It contains moreover the essence of the main idea for proving Theorem 3.3.

Section 5 describes the proof of Theorem 3.3.

Section 6 is devoted to the proof of Theorem 1.5.

Section 7 contains a few final comments and remarks.

2 Definitions

All facts concerning lattices needed in the sequel are collected in this Section for the convenience of the reader, see [5] and [9] for more on lattices and lattice-packings.

An $n-$*dimensional lattice* is a discret-cocompact subgroup Λ of the $n-$dimensional Euclidean vector space $\mathbf{E}^n$. Denoting by $\langle\ ,\ \rangle$ the scalar product and choosing a $\mathbf{Z}-$basis $b_1,\dots,b_n$ of a lattice $\Lambda=\oplus_{j=1}^n\mathbf{Z}b_j$, the positive definite symmetric matrix $G\in\mathbf{R}^{n\times n}$ with coefficients

$$G_{i,j}=\langle b_i,b_j\rangle$$

is a *Gram matrix* of Λ. Its determinant $\det(G)$, called the *determinant* of Λ, is independent of the choosen basis $b_1,\dots b_n$ and equals the squared volume of the flat torus $\mathbf{E}^n/\Lambda$. The *norm* of a lattice vector $\lambda\in\Lambda$ is defined as $\langle\lambda,\lambda\rangle$ and equals thus the square of the Euclidean norm $\sqrt{\langle\lambda,\lambda\rangle}$. A lattice Λ is *integral* if all scalar products $\{\langle\lambda,\mu\rangle\mid\lambda,\mu\in\Lambda\}$ are integral. An integral lattice of determinant 1 is *unimodular*. An Euclidean lattice Λ is unimodular if and only if every group homomorphism $\varphi:\Lambda\longrightarrow\mathbf{Z}$ is of the form $\varphi(v)=\langle v,w_\varphi\rangle$ for a suitable fixed element $w_\varphi\in\Lambda$. The *minimum*

$$\min\Lambda\ =\min_{\lambda\in\Lambda\setminus\{0\}}\langle\lambda,\lambda\rangle$$

of a lattice Λ is the norm of a shortest non-zero vector in Λ. The *density* $\Delta(\Lambda)$ and the *center-density* $\delta(\Lambda)$ of an $n-$dimensional lattice Λ are defined as

$$\Delta(\Lambda)=\sqrt{\frac{(\min\Lambda)^n}{4^n\det\Lambda}}\,V_n\qquad\text{and}\qquad\delta(\Lambda)=\sqrt{\frac{(\min\Lambda)^n}{4^n\det\Lambda}}$$

where $V_n=\pi^{n/2}/(n/2)!$ denotes the volume of the $n-$dimensional unit-ball in $\mathbf{E}^n$. These two densities are proportional for a given fixed dimension n and $\Delta(\Lambda)$ measures the (asymptotic) proportion of space occupied by the *sphere packing of* Λ obtained by centering $n-$dimensional Euclidean balls of radius $\sqrt{\min\Lambda/4}$ at all points of Λ.

Given an $n-$dimensional lattice $\Lambda\subset\mathbf{E}^n$ the subset

$$\Lambda^\sharp=\{x\in\mathbf{E}^n\mid\langle x,\lambda\rangle\in\mathbf{Z}\quad\forall\lambda\in\Lambda\}$$

is also a lattice called the *dual lattice* of Λ. The scalar product induces a natural bijection between $\Lambda^\sharp$ and the set of homomorphisms $\Lambda\longrightarrow\mathbf{Z}$. A lattice Λ is integral if and only if $\Lambda\subset\Lambda^\sharp$. For an integral lattice, the *determinant group* $\Lambda^\sharp/\Lambda$ is a finite abelian group consisting of $(\det\Lambda)$ elements.

A sublattice $M\subset\Lambda$ is *saturated* if Λ/M is without torsion or equivalently if $M=(M\otimes_{\mathbf{Z}}\mathbf{R})\cap\Lambda$. A sublattice $\Lambda\subset M$ is thus saturated if and only if M is a direct factor of the additive group M.

The following result is well-known (cf. Chapter I, Proposition 9.8 in [9]):

Proposition 2.1 *Let M and N be two saturated sublattices of dimension m and n in a common unimodular lattice Λ of dimension $m+n$ such that M and N are contained in orthogonal subspaces.*

Then the two determinant groups $M^\sharp/M$ and $N^\sharp/N$ are isomorphic. In particular, the lattices M and N have equal determinants.

Proof Since $\Lambda = \Lambda^\sharp$ is unimodular, orthogonal projection $\Lambda \longrightarrow M^\sharp$ yields a surjective homomorphism from Λ onto $M^\sharp$ with kernel $(N \otimes_{\mathbf{Z}} \mathbf{R}) \cap \Lambda$ coinciding with N since N is saturated. This shows $M^\sharp \sim \Lambda/N$ and thus $M^\sharp/M \sim \Lambda/(M \oplus N)$. Exchanging the role of M and N implies the result. $\square$

Two lattices Λ and M are *similar*, if there exists a bijection $\Lambda \longrightarrow M$ which extends to an Euclidean similarity from $\Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ to $M \otimes_{\mathbf{Z}} \mathbf{R}$. The set of similarity classes of lattices is endowed with a natural topology: a neighbourhood of an $n-$dimensional lattice Λ is given by all lattices having a Gram matrix in $\mathbf{R}_{>0}\ V(G)$ where $V(G) \subset \mathbf{R}^{n\times n}$ is a neighbourhood of a fixed Gram matrix G of Λ.

Similar lattices have identical densities and the density function $\Lambda \longmapsto \Delta(\Lambda)$ is continuous with respect to the natural topology on similarity classes.

Consider the set $\mathcal{L}_n = \{\Lambda(s) \mid s \in \mathbf{N}^{n+1} \setminus \{0\}\}$ of all $n-$dimensional integral lattices of the form

$$\Lambda(s) = \{z \in \mathbf{Z}^{n+1} \mid \langle z, s\rangle = 0\}$$

where $s \in \mathbf{N}^{n+1} \setminus \{0\}$ is a non-zero vector of length $n+1$ with non-negative integral coordinates.

Proposition 2.2 *The set $\mathcal{L}_n$ is dense in the set of similarity classes of $n-$dimensional Euclidean lattices.*

There are thus lattices in $\mathcal{L}_n$ with densities arbitrarily close to the maximal density Δ_n of $n-$dimensional lattices.

Proof of Proposition 2.2 Given a Gram matrix $G = \langle b_i, b_j\rangle$ of an $n-$dimensional lattice $\Lambda = \oplus_{j=1}^n \mathbf{Z} b_j$, Gram-Schmidt orthogonalization of the $\mathbf{Z}-$basis $b_1, \dots, b_n$ (with respect to the Euclidean scalar product) yields a matrix factorization

$$G = L\ L^t$$

where $L = (l_{i,j})_{1 \le i,j \le n}$ is an invertible lower triangular matrix.

Choose a large real number $\kappa > 0$ and consider the integral lower triangular matrix $\tilde{L}(\kappa)$ whose coefficients $\tilde{l}_{i,j} \in \mathbf{Z}$ satisfy

$$|\tilde{l}_{i,j} - \kappa l_{i,j}| \le 1/2$$

and are obtained by rounding off each coefficient of κL to a nearest integer.

Define the integral matrix

$$B(\kappa) = \begin{pmatrix} \tilde{l}_{1,1} & 1 & 0 & 0 & \dots \\ \tilde{l}_{2,1} & \tilde{l}_{2,2} & 1 & 0 & \\ \vdots & & \ddots & \ddots & \\ \tilde{l}_{n,1} & \tilde{l}_{n,2} & \dots & \tilde{l}_{n,n} & 1 \end{pmatrix}$$

of size $n \times (n+1)$ with coefficients

$$b_{i,j} = \begin{cases} \tilde{l}_{i,j} & \text{if } j \leq i \\ 1 & \text{if } j = i+1 \\ 0 & \text{otherwise .} \end{cases}$$

The rows of $B(\kappa)$ span an integral sublattice $\tilde{\Lambda}(\kappa)$ of dimension n in $\mathbf{Z}^{n+1}$. Moreover, the lattice $\tilde{\Lambda}(\kappa)$ is saturated since deleting the first column of $B(\kappa)$ yields an integral unimodular square matrix of size $n \times n$. The special form of $B(\kappa)$ shows that there exists an integral column-vector

$$v(\kappa) = \begin{pmatrix} 1 \\ -\tilde{l}_{1,1} \\ \tilde{l}_{1,1}\tilde{l}_{2,2} - \tilde{l}_{2,1} \\ \vdots \end{pmatrix} \in \mathbf{Z}^{n+1}$$

such that $B(\kappa)v(\kappa) = 0$. We have thus

$$\tilde{\Lambda}(\kappa) = v(\kappa)^{\perp} \cap \mathbf{Z}^{n+1} \subset \mathbf{E}^{n+1} .$$

Since $\lim_{\kappa\to\infty}\frac{1}{\kappa}\ B(\kappa)$ is given by the matrix L with an extra column of zeros appended, we have

$$\lim_{\kappa\to\infty}\frac{1}{\kappa^2}\ B(\kappa)(B(\kappa))^t = G$$

and the lattice $\frac{1}{\kappa}\tilde{\Lambda}(\kappa)$ converges thus to the lattice Λ for $\kappa \to \infty$. Considering the integral vector $s = (s_0, s_1, \ldots) \in \mathbf{N}^{n+1}$ defined by $s_i = |v(\kappa)_{i+1}|$ for $i = 0, \ldots, n$, we get an integral lattice

$$\{z = (z_0, \ldots, z_n) \in \mathbf{Z}^{n+1} \mid \langle z, s\rangle = 0\}$$

of $\mathcal{L}_n$ which is isometric to $\tilde{\Lambda}(\kappa)$. $\square$

3 $\mu-$sequences

Let $\mu \geq 2$ be a strictly positive integer. A $\mu-$*sequence* is a finite or infinite sequence $s_0 = 1, s_1, s_2, \ldots$ of $(l+1)$ strictly positive integers such that the $n-$dimensional lattice

$$\Lambda_n = \{(z_0, z_1, \ldots, z_n) \in \mathbf{Z}^{n+1} \mid \sum_{k=0}^{n} s_k z_k = 0\} = (s_0, \ldots, s_n)^{\perp} \cap \mathbf{Z}^{n+1}$$

has minimum $\geq \mu$ for all $n \geq 1$ which make sense (ie. for $n \leq l$ if the sequence $(s_0, s_1, \ldots, s_l)$ has finite length l). The letter μ in a $\mu-$sequence

stands for minimum and should not be confused with the Moebius-function, (unfortunately also) denoted $\mu : \mathbf{N} \longrightarrow \{-1, 0, 1\}$.

Since Λ_n is saturated in $\mathbf{Z}^{n+1}$ by construction and orthogonal to the $1-$dimensional saturated lattice $\mathbf{Z}(s_0, \dots, s_n) \subset \mathbf{Z}^{n+1}$, Proposition 2.1 shows that we have $\det(\Lambda_n) = \sum_{k=0}^n s_k^2$. We get thus a lower bound for the density

$$\Delta(\Lambda_n) = \sqrt{\frac{(\min\ \Lambda_n)^n}{4^n\ \det\ \Lambda_n}} V_n \geq \sqrt{\frac{\mu^n}{4^n\ \sum_{k=0}^n s_k^2}} V_n$$

of the $n-$dimensional lattice Λ_n associated to a $\mu-$sequence $(s_0, \dots, s_n, \dots)$. This lower bound is an equality except if the sequence $(s_0, \dots, s_n)$ is a $(\mu + 1)-$sequence.

Remark 3.1 *We hope that the double meaning of μ will not confuse the reader: $\mu(l) \in \{-1, 0, 1\}$ denotes always the Möbius function of a natural integer l while μ or $\mu_1, \mu_2, \dots$ stands for natural integers.*

Remark 3.2 *(i) The condition $s_0 = 1$ ensures that $(s_0, \dots, s_n) \subset \mathbf{Z}^{n+1}$ generates a saturated $1-$dimensional sublattice and will be useful for proving Lemma 5.1. It can however be weakened by requiring that $s_0, \dots, s_n$ are without common non-trivial divisor. Lemma 5.1 (which applies to a sequence of $\mu-$sequences) remains valid if s_0 is uniformly bounded.*

It is of course also possible (but not very useful) to consider sequences with coefficients in $\mathbf{Z}$.

(ii) Any subsequence $s_{i_0} = s_0, s_{i_1}, s_{i_2}, \dots$ of a $\mu-$sequence is again a $\mu-$sequence and permuting the terms of a $\mu-$sequence by a permutation fixing s_0 yields of course again a $\mu-$sequence.

(iii) Lattices associated to $\mu-$sequences are generally neither perfect nor eutactic (cf. [9] for definitions). Their densities can thus generally be improved by suitable deformations.

Theorem 3.3 *Let $\mu_1, \mu_2, \dots$ be a strictly increasing sequence of natural integers $2 \leq \mu_1 < \mu_2 < \dots$. Suppose that we have finite μ_r-sequences $(s(\mu_r)_0, \dots, s(\mu_r)_{n-1})$ with existing limit-density*

$$\tilde{\Delta}_{n-1} = lim_{r \to \infty} \frac{\mu_r^{(n-1)/2}}{\sqrt{4^{n-1} \sum_{i=0}^{n-1} s(\mu_r)_i^2}} V_{n-1} > 0$$

for the sequence of lattices $(s(\mu_r)_0, \dots, s(\mu_r)_{n-1})^\perp \subset \mathbf{Z}^n$.

Let σ_n be a positive real number such that

$$2^{n-1} \tilde{\Delta}_{n-1} \sum_{k=1}^{A} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \left(1 - k^2 \left(2^{n-1} \tilde{\Delta}_{n-1} \frac{V_n}{V_{n-1}} \sigma_n \right)^2 \right)^{(n-1)/2} < 1$$

where

$$A = \left\lfloor \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \sigma_n} \right\rfloor .$$

Then there exists a natural integer R such that for all $r \geq R$, the μ_r-sequence $(s(\mu_r)_0, \ldots, s(\mu_r)_{n-1})$ can be extended to a μ_r-sequence $(s(\mu_r)_0, \ldots, s(\mu_r)_n)$ satisfying $0 < s(\mu_r)_n < \sigma_n \mu_r^{n/2} V_n$.

The proof of Theorem 3.3 will be given in section 5. Assuming Theorem 3.3, we proceed now to prove Theorem 1.1.

3.1 Proof of Theorem 1.1 and 1.2

Suppose that the inequality of assertion (i) does not hold for some natural integer n. By Proposition 2.2 we can find a sequence of finite μ_r-sequences $(s(\mu_r)_0, \ldots, s(\mu_r)_{n-1})$ (with $\mu_r \to \infty$) such that

$$\lim_{r\to\infty} \frac{\mu_r^{(n-1)/2}}{\sqrt{4^{n-1} \sum_{i=0}^{n-1} s(\mu_r)_i^2}} V_{n-1} = \Delta_{n-1} .$$

Consider the function

$$x \longmapsto 2^{n-1} \Delta_{n-1} \sum_{k=1}^{A(x)} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \left(1 - k^2 \left(2^{n-1} \Delta_{n-1} \frac{V_n}{V_{n-1}} x \right)^2 \right)^{(n-1)/2}$$

where $A(x) = \lfloor \frac{2^{1-n} V_{n-1}}{V_n \Delta_{n-1} x} \rfloor$ for $x > 0$. We claim that this function is continuous and strictly decreasing in x. Indeed, for increasing $x \in [0, \frac{2^{1-n} V_{n-1}}{k V_n \Delta_{n-1}}]$, a summand

$$\left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \left(1 - k^2 \left(2^{n-1} \Delta_{n-1} \frac{V_n}{V_{n-1}} x \right)^2 \right)^{(n-1)/2}$$

decreases continuously from $\sum_{l|k} \frac{\mu(l)}{l^{n-1}} = \prod_{p|k} \left(1 - \frac{1}{p^{n-1}} \right) > 0$ to 0. Such a summand disappears if it becomes zero and their number $A(x) = \lfloor \frac{2^{1-n} V_{n-1}}{V_n \Delta_{n-1} x} \rfloor$ is finite and decreases with x.

This shows that we can choose a positive real number $\sigma_n < \frac{1}{2^n \Delta_n}$ such that we have the inequalities

$$2^{n-1} \Delta_{n-1} \sum_{k=1}^{\lfloor 2\Delta_n V_{n-1}/(\Delta_{n-1} V_n) \rfloor} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \left(1 - \left(\frac{k \Delta_{n-1} V_n}{2 \Delta_n V_{n-1}} \right)^2 \right)^{(n-1)/2} <$$

$$< 2^{n-1} \Delta_{n-1} \sum_{k=1}^{A} \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}} \right) \left(1 - k^2 \left(2^{n-1} \Delta_{n-1} \frac{V_n}{V_{n-1}} \sigma_n \right)^2 \right)^{(n-1)/2} < 1$$

where $A = \lfloor \frac{2^{1-n}V_{n-1}}{V_n \Delta_{n-1}\sigma_n} \rfloor$.

Applying Theorem 3.3 and extracting a suitable subsequence from $\mu_1, \mu_2, \ldots$, we can suppose that all sequences $(s(\mu_r)_0, \ldots, s(\mu_r)_{n-1})$ can be extended to μ_r-sequences $(s(\mu_r)_0, \ldots, s(\mu_r)_n)$ with

$$\lim_{r\to\infty} \frac{s(\mu_r)_n}{\mu_r^{n/2}} = \alpha \leq \sigma_n V_n \ .$$

Using

$$\lim_{r\to\infty} \frac{1}{\mu_r^{n-1}} \sum_{i=0}^{n-1} s(\mu_r)_i^2 = \frac{V_{n-1}^2}{4^{n-1}\Delta_{n-1}^2}$$

we have

$$\lim_{r\to\infty} \frac{1}{\mu_r^n} \sum_{i=0}^{n} s(\mu_r)_i^2 = \lim_{r\to\infty} \frac{1}{\mu_r} \ \frac{V_{n-1}^2}{4^{n-1}\Delta_{n-1}^2} + \alpha^2 = \alpha^2$$

and get the existence of a sequence of $n-$dimensional lattices

$$(s(\mu_r)_0, \ldots, s(\mu_r)_n)^\perp \subset \mathbf{Z}^{n+1}$$

with limit-density

$$\lim_{r\to\infty} \sqrt{\frac{\mu_r^n}{4^n \sum_{i=0}^n s(\mu_r)_i^2}} V_n = \frac{1}{2^n \alpha} V_n \geq \frac{1}{2^n \sigma_n V_n} V_n > \Delta_n$$

in contradiction with maximality of Δ_n. □

Proof of Theorem 1.2 Choose $\xi < x$ such that $C_n(x) = \xi\ Y_n(\xi)^{2/n}$. The inequality $\xi \leq \gamma_{n-1}$ implies the existence of an $(n-1)$–dimensional lattice with density $2^{-(n-1)}\xi^{(n-1)/2}V_{n-1}$. Theorem 3.3 implies the existence of $n-$dimensional lattice with density $2^{-n}(C_n(x))^{n/2}V_n$. □

4 An easy crude bound for the lexicographically first $\mu-$sequence

Theorem 4.1 *Given an integer $\mu \geq 2$ there exists an increasing $\mu-$sequence $s_0 = 1 \leq s_1 \leq \ldots$ such that*

$$s_n \leq 1 + \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} \leq \sqrt{\mu}\sqrt{\mu + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!}$$

for all $n \geq 1$.

The proof of Theorem 4.1 is elementary and consists of an analysis of the "greedy algorithm" which constructs the first $\mu-$sequence with respect to the lexicographic order on sequences. An easy analysis shows that the

lexicographically first $\mu-$sequence satisfies the first inequalities of Theorem 4.1. The greedy algorithm, although very simple, is unfortunately useless for practical purposes.

A $\mu-$sequence satisfying the inequalities of Theorem 4.1 yields already rather dense lattices as shown by the next result.

Corollary 4.2 *For any $\mu \geq 2$, there exists a $\mu-$sequence $(s_0, s_1, \ldots, s_n) \in \mathbf{Z}^{n+1}$ such that the density of the associated lattice $\Lambda_n = (s_0, \ldots, s_n)^\perp \cap \mathbf{Z}^{n+1}$ satisfies*

$$\Delta(\Lambda_n) \geq \frac{(1 + n/(4\mu))^{-n/2}}{2^n \sqrt{(n+1)\mu}} \ .$$

Remark 4.3 *Taking $\mu \sim n^2/4$ we get for large n the existence of $n-$dimensional lattices with density Δ roughly at least equal to*

$$\frac{1}{2^{n-1}\ n\ \sqrt{(n+1)\ e}}$$

which is reasonably close to the Minkowski-Hlawka bound $\Delta_n \geq \zeta(n)\ 2^{1-n}$.

Lemma 4.4 *The standard Euclidean lattice $\mathbf{Z}^n$ contains at most*

$$2\sqrt{\mu + n/4}^{\,n}\ \frac{\pi^{n/2}}{(n/2)!}$$

vectors of (squared Euclidean) norm $\leq \mu$.

Proof We denote by

$$B_{\leq\sqrt{\rho}}(x) = \{z \in \mathbf{E}^n \mid \langle z - x, z - x\rangle \leq \rho\}$$

the closed Euclidean ball with radius $\sqrt{\rho} \geq 0$ and center $x \in \mathbf{E}^n$. Given $\sqrt{\mu}, \sqrt{\rho} \geq 0$ and $x \in B_{\leq\sqrt{\mu}}(0)$, the closed half-ball

$$\{z \in \mathbf{E}^n \mid \langle z, x\rangle \leq \langle x, x\rangle\} \cap B_{\leq\sqrt{\rho}}(x)$$

(obtained by intersecting the closed affine halfspace $H_x = \{z \in \mathbf{E}^n \mid \langle z, x\rangle \leq \langle x, x\rangle\}$ with the Euclidean ball $B_{\leq\sqrt{\rho}}(x)$ centered at $x \in \partial H_x$) is contained in $B_{\leq\sqrt{\mu+\rho}}(0)$.

Since the regular standard cube

$$C = [-\frac{1}{2}, \frac{1}{2}]^n \subset \mathbf{E}^n$$

of volume 1 is contained in a ball of radius $\sqrt{n/4}$ centered at the origin, the intersection

$$(z + C) \cap \{x \in \mathbf{E}^n \mid \langle x, x\rangle \leq \mu + n/4\} = (z + C) \cap B_{\leq\sqrt{\mu+n/4}}(0)$$

is of volume at least $1/2$ for any element $z \in \mathbf{E}^n$ of norm $\langle z, z\rangle \leq \mu$.

Since integral translates of C tile $\mathbf{E}^n$, we have

$$\frac{1}{2}\sharp\{z \in \mathbf{Z}^n \mid \langle z, z\rangle \leq \mu\} \leq \mathrm{Vol}\ \{x \in \mathbf{E}^n \mid \langle x, x\rangle \leq \mu + n/4\}\ .$$

Using the fact that the unit ball in Euclidean $n-$space has volume $\pi^{n/2}/(n/2)!$ (cf. Chapter 1, formula 17 in [5]) we get the result. $\square$

Proof of Theorem 4.1 For $n = 0$, the first inequality boils down to $s_0 = 1 \leq 1 + \sqrt{\mu - 2}$ and holds for $\mu \geq 2$. Consider now for $n \geq 1$ a $\mu-$sequence $(s_0, \dots, s_{n-1}) \in \mathbf{N}^n$.

Introduce the set

$$\mathcal{F}_n = \{(a, k) \in \mathbf{N}^2 \mid \exists\ z = (z_0, \dots, z_{n-1}) \in \mathbf{Z}^n \setminus \{0\} \text{ such that}$$
$$ak = \langle z, (s_0, \dots, s_{n-1})\rangle \text{ and } \langle z, z\rangle + k^2 < \mu\}\ .$$

Since Λ_{n-1} has minimum $\geq \mu$, the equality $\langle z, (s_0, \dots, s_{n-1})\rangle = 0$ implies $\langle z, z\rangle \geq \mu$ for $z \in \mathbf{Z}^n \setminus \{0\}$. This shows that we have $a, k > 0$ for $(a, k) \in \mathcal{F}_n$.

Since for a given pair of opposite non-zero vectors $\pm z \in \mathbf{Z}^n$ with norm $0 < \langle z, z\rangle < \mu$ there are at most $\sqrt{\mu - 1 - \langle z, z\rangle} \leq \sqrt{\mu - 2}$ strictly positive integers k such that $\langle z, z\rangle + k^2 < \mu$, such a pair $\pm z$ of vectors contributes at most $\sqrt{\mu - 2}$ distinct elements to $\mathcal{F}_n$. The cardinality $f_n = \sharp(\mathcal{F}_n)$ of $\mathcal{F}_n$ is thus bounded by

$$f_n \leq \sqrt{\mu - 2}\frac{\sharp\{z \in \mathbf{Z}^n \mid 0 < \langle z, z\rangle \leq \mu - 1\}}{2} \leq \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^n\,\frac{\pi^{n/2}}{(n/2)!}$$

where the last inequality follows from Lemma 4.4. There exists thus a strictly positive integer

$$s_n \leq f_n + 1 \leq 1 + \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^n\,\frac{\pi^{n/2}}{(n/2)!}$$

such that $(s_n, k) \notin \mathcal{F}_n$ for all $k \in \mathbf{N}$. The strictly positive integer s_n satisfies the first inequality of Theorem 4.1 and it is straightforward to check that the $n-$dimensional lattice

$$\Lambda_n = \{z \in \mathbf{Z}^{n+1} \mid \sum_{i=0}^{n} s_i z_i = 0\}$$

has minimum $\geq \mu$. This shows the first inequality. Choosing for s_n the smallest strictly positive integer such that $(s_n, k) \notin \mathcal{F}_n$ for all $k \in \mathbf{N}$ and iterating this construction yields clearly an increasing $\mu-$sequence.

The second inequality

$$1 + \sqrt{\mu - 2}\sqrt{\mu - 1 + n/4}^n\,\frac{\sqrt{\pi}^n}{(n/2)!} \leq \sqrt{\mu}\sqrt{\mu + n/4}^n\,\frac{\sqrt{\pi}^n}{(n/2)!}$$

of Theorem 4.1 boils down to

$$1 \leq \sqrt{2}\sqrt{2+n/4}^n \ \frac{\sqrt{\pi}^n}{(n/2)!}$$

for $\mu = 2$. This inequality is clearly true since the $n-$dimensional Euclidean ball of radius $\sqrt{2+n/4}$ has volume $\sqrt{2+n/4}^n \ \frac{\sqrt{\pi}^n}{(n/2)!}$ and contains the regular cube $[-\frac{1}{2}, \frac{1}{2}]^n$ of volume 1.

For $\mu \geq 3$ we have to establish the inequality $\Phi(1) - \Phi(0) \geq 1$ where

$$\Phi(t) = \sqrt{\mu - 2 + 2t}\sqrt{\mu - 1 + t + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} \ .$$

We get thus

$$\begin{array}{rcl} \Phi(1) - \Phi(0) & \geq & \inf_{\xi \in (0,1)} \Phi'(\xi) \\ & \geq & \frac{1}{\sqrt{\mu}}\sqrt{\mu - 1 + n/4}^n \frac{\sqrt{\pi}^n}{(n/2)!} + \frac{n}{2}\sqrt{\mu-2}\sqrt{\mu - 1 + n/4}^{n-2} \frac{\sqrt{\pi}^n}{(n/2)!} \ . \end{array}$$

For $n = 1$ and $\mu \geq 2$ we have

$$\Phi(1) - \Phi(0) \geq \sqrt{1 - \frac{1}{\mu}} \frac{\sqrt{\pi}}{\sqrt{\pi}/2} \geq \frac{2}{\sqrt{2}} > 1 \ .$$

For $n \geq 2$ and $\mu \geq 3$ we get

$$\Phi(1) - \Phi(0) \geq \sqrt{2 + n/4}^{n-2} \frac{\sqrt{\pi}^{n-2}}{((n-2)/2)!} \ \pi$$

and the right-hand side equals $\pi > 1$ for $n = 2$. For $n > 2$, the right hand side equals π times the volume of the $(n-2)-$dimensional ball of radius $\sqrt{2+n/4}$ containing the regular cube $[-\frac{1}{2}, \frac{1}{2}]^{n-2}$ of volume 1. The second inequality follows. □

Proof of Corollary 4.2 Theorem 4.1 shows the existence of a $\mu-$sequence $(s_0 = 1, \ldots, s_n)$ satisfying

$$s_0, \ldots, s_n \leq \sqrt{\mu}\sqrt{\mu + n/4}^n \ \frac{\sqrt{\pi}^n}{(n/2)!} \ .$$

This shows for the lattice $\Lambda_n = (s_0, \ldots, s_n)^\perp \cap \mathbf{Z}^{n+1}$ the inequality

$$\det \Lambda_n = \sum_{i=0}^n s_i^2 \leq (n+1)\mu(\mu + n/4)^n \frac{\pi^n}{((n/2)!)^2} = (n+1)\mu(\mu + n/4)^n V_n^2$$

and implies

$$\Delta(\Lambda_n) \geq \sqrt{\frac{\mu^n}{4^n (n+1)\mu(\mu+n/4)^n V_n^2}} V_n$$

which proves Corollary 4.2. □

5 Proof of Theorem 3.3

The main idea for proving Theorem 3.3 is to get rid of a factor $\sqrt{\mu}$ when computing an upper bound f for the size of the finite set $\mathcal{F}$ considered in the proof of Theorem 4.1. This is possible since the volume of the standard unit-ball of large dimension concentrates along linear hyperplanes. During the proof, we use for simplicity the slightly abusive notation $\mu = \mu_k$ and $(s_0, \ldots, s_n) = (s(\mu_k)_0, \ldots, s(\mu_k)_n)$. Since μ belongs to the strictly increasing integral sequence $\mu_1 < \mu_2 < \ldots$ tending to infinity, we consider sequences in the $\mu \to \infty$ limit. This allows us to neglect boundary effects when replacing counting arguments by volume-computations.

In the sequel we write

$$g(x) \sim_{x\to\alpha} h(x) \ , \ \text{respectively } g(x) \leq_{x\to\alpha} h(x) \ ,$$

for

$$\lim_{x\to\alpha}\frac{g(x)}{h(x)} = 1 \ , \ \text{respectively } \operatorname{limsup}_{x\to\alpha}\frac{g(x)}{h(x)} \leq 1 \ ,$$

where $g(x),\ h(x) > 0$.

Proof of Theorem 3.3 We prove first a weaker statement assuming the stronger inequality

$$2^{n-1}\tilde{\Delta}_{n-1}\sum_{k=1}^{A}\left(1-k^2\left(2^{n-1}\tilde{\Delta}_{n-1}\frac{V_n}{V_{n-1}}\sigma_n\right)^2\right)^{(n-1)/2} < 1$$

where

$$A = \lfloor \frac{2^{1-n}V_{n-1}}{V_n\tilde{\Delta}_{n-1}\sigma_n}\rfloor.$$

Details for dealing with the extra factor $\left(\sum_{l|k}\frac{\mu(l)}{l^{n-1}}\right)$ will be given later.

Choose a positive real number $\tilde{\sigma}_n < \sigma_n$ such that we have the equalities

$$A = \lfloor \frac{2^{1-n}V_{n-1}}{V_n\tilde{\Delta}_{n-1}\sigma_n}\rfloor = \lfloor \frac{2^{1-n}V_{n-1}}{V_n\tilde{\Delta}_{n-1}\tilde{\sigma}_n}\rfloor$$

(where $\lfloor x\rfloor \in \mathbf{Z}$ denotes the integer part of $x \in \mathbf{R}$) and the inequality

$$2^{n-1}\tilde{\Delta}_{n-1}\sum_{k=1}^{A}\left(1-k^2\left(2^{n-1}\tilde{\Delta}_{n-1}\frac{V_n}{V_{n-1}}\tilde{\sigma}_n\right)^2\right)^{(n-1)/2} < 1.$$

We fix $\tilde{\sigma}_n$ in the sequel and introduce $\epsilon = \frac{\sigma_n}{\tilde{\sigma}_n} - 1 > 0$. We prove Theorem 3.3 for all μ huge enough by showing the existence of a μ−sequence $(s_0, \ldots, s_{n-1}, s_n)$ with $s_n \in I \cap \mathbf{N}$ where

$$I = [\tilde{\sigma}_n\mu^{n/2}V_n, (1+\epsilon)\tilde{\sigma}_n\mu^{n/2}V_n] = [\tilde{\sigma}_n\mu^{n/2}V_n, \sigma_n\mu^{n/2}V_n] \ .$$

Since our computations rely on strict inequalities involving volume-computations which are continuous in $\tilde{\Delta}_{n-1}$, this will imply the weakened form (without the factor $\left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right)$) of Theorem 3.3.

For $k = 1, 2, \ldots \in \mathbf{N}$ we define finite subsets

$$I_k = \{s \in I \cap \mathbf{N} | \sum_{i=0}^{n-1} s_i x_i = ks \text{ for some } (x_0, \ldots, x_{n-1}) \in B_{<\sqrt{\mu-k^2}} \cap \mathbf{Z}^n\}$$

of natural integers in $I \cap \mathbf{N}$ where $B_{<\sqrt{\mu-k^2}} \cap \mathbf{Z}^n$ denotes the set of all integral vectors $(x_0, \ldots, x_{n-1}) \in \mathbf{Z}^n$ having (squared Euclidean) norm strictly smaller than $\mu - k^2$.

We have

$$\begin{aligned} |\sum_{i=0}^{n-1} s_i x_i| \leq \sqrt{\sum_{i=0}^{n-1} s_i^2} \sqrt{\sum_{i=0}^{n-1} x_i^2} &\leq_{\mu\to\infty} \sqrt{\frac{\mu^{n-1} V_{n-1}^2}{4^{n-1} \tilde{\Delta}_{n-1}^2}} \sqrt{\mu - k^2} \\ &\sim_{\mu\to\infty} \frac{2^{1-n}\ \mu^{n/2} V_{n-1}}{\tilde{\Delta}_{n-1}} \end{aligned}$$

for $(x_0, \ldots, x_{n-1}) \in B_{<\sqrt{\mu-k^2}}$. This shows $I_k = \{\emptyset\}$ if

$$k \geq A + 1 > \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \tilde{\sigma}_n} .$$

An extension $(s_0, \ldots, s_{n-1}, s_n)$ with $s_n \in I$ of a $\mu-$sequence $(s_0, \ldots, s_{n-1})$ is a $\mu-$sequence if and only if $s_n \notin \bigcup_{k=1}^{A} I_k$.

Introducing the sets

$$X_k(a) = \{(x_0, \ldots, x_{n-1}) \in \mathbf{Z}^n \mid \sum_{i=0}^{n-1} s_i x_i \in kI \cap (k\mathbf{N} + a),\ \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\},$$

we have obviously $\sharp(I_k) \leq \sharp(X_k(0))$. This ensures the existence of a $\mu-$sequence $(s_0, \ldots, s_{n-1}, s_n)$ with $s_n \in I \cap \mathbf{N}$ if we have

$$\sum_{k=1}^{A} \sharp(X_k(0)) < \sharp\{I \cap \mathbf{N}\}. \tag{1}$$

Denoting by

$$X_k(*) = \{(x_0, \ldots, x_{n-1}) \in \mathbf{Z}^n \mid \frac{1}{k} \sum_{i=0}^{n-1} s_i x_i \in I,\ \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\}$$

the union of the disjoint sets $X_k(0), X_k(1), \ldots, X_k(k-1)$, the following asymptotic equalities hold.

Lemma 5.1 *We have*

$$\sharp(X_k(j)) \sim_{\mu\to\infty} \frac{1}{k}\sharp(X_k(*))$$

for $j = 0, \ldots, k-1$.

It is thus enough to compute $\sharp(X_k(*))$ in order to get an asymptotic estimation of $X_k(0) \sim_{\mu\to\infty} \frac{1}{k}\sharp(X_k(*))$. We have

$$\begin{aligned}\sharp(X_k(*)) &= \sharp\{(x_0, \ldots, x_{n-1}) \in \mathbf{Z}^n \mid \frac{1}{k}\sum_{i=0}^{n-1} s_i x_i \in I,\ \sum_{i=0}^{n-1} x_i^2 < \mu - k^2\} \\ &\sim_{\mu\to\infty} \mathrm{Vol}\{(t_0, \ldots, t_{n-1}) \in \mathbf{E}^n \mid \sum_{i=0}^{n-1} t_i^2 \le \mu,\ \frac{1}{k}\sum_{i=0}^{n-1} s_i t_i \in I\}\end{aligned}$$

and the requirement $\frac{1}{k}\sum_{i=0}^{n-1} s_i t_i \in I$ amounts to the inequalities

$$k\tilde{\sigma}_n \mu^{n/2} V_n \le \sum s_i t_i \le k\sigma_n \mu^{n/2} V_n.$$

For huge μ (and fixed k), the number $k\sharp(X_k)$ is thus essentially the volume W_k of a subset of the $n-$dimensional ball of radius $\sqrt{\mu}$. More precisely, this subset is delimited by the two parallel affine hyperplanes orthogonal to $(s_0, \ldots, s_{n-1})$ which are at distance

$$k\tilde{D} = k\frac{\tilde{\sigma}_n \mu^{n/2} V_n}{\sqrt{\sum_{i=0}^n s_i^2}} \sim_{\mu\to\infty} k\sqrt{\mu}\ 2^{n-1}\frac{V_n}{V_{n-1}}\tilde{\Delta}_{n-1}\tilde{\sigma}_n$$

and $(1+\epsilon)k\tilde{D}$ of the origin.

We have thus

$$\begin{aligned} W_k &= \int_{k\tilde{D}}^{k(1+\epsilon)\tilde{D}} \left(\mu - t^2\right)^{(n-1)/2} dt V_{n-1} \le \epsilon k\tilde{D}\left(\mu - k^2\tilde{D}^2\right)^{(n-1)/2} V_{n-1} \\ &\le_{\mu\to\infty} \epsilon k\tilde{\sigma}_n \mu^{n/2}\ 2^{n-1} V_n \tilde{\Delta}_{n-1}\left(1 - k^2\left(2^{n-1}\tilde{\sigma}_n \tfrac{V_n}{V_{n-1}}\tilde{\Delta}_{n-1}\right)^2\right)^{(n-1)/2}. \end{aligned}$$

Using the asymptotic equalities $\sharp(X_k) \sim_{\mu\to\infty} \frac{W_k}{k}$, we get

$$\sum_{k=1}^{A}\sharp(X_k) \le_{\mu\to\infty} \epsilon\tilde{\sigma}_n \mu^{n/2}\ 2^{n-1}\tilde{\Delta}_{n-1} V_n \sum_{k=1}^{A}\left(1 - k^2\left(2^{n-1}\tilde{\sigma}_n\tilde{\Delta}_{n-1}\frac{V_n}{V_{n-1}}\right)^2\right)^{(n-1)/2}.$$

Together with the obvious estimation

$$\sharp\{I \cap \mathbf{N}\} \sim_{\mu\to\infty} \epsilon\tilde{\sigma}_n \mu^{n/2} V_n,$$

we have now

$$\frac{\sharp\{I \cap \mathbf{N}\}}{\sum_{k=1}^{A}\sharp(I_k)} \ge_{\mu\to\infty} \frac{2^{1-n}}{\tilde{\Delta}_{n-1}\sum_{k=1}^{A}\left(1 - k^2\left(2^{n-1}\tilde{\sigma}_n\tilde{\Delta}_{n-1}\frac{V_n}{V_{n-1}}\right)^2\right)^{(n-1)/2}} > 1$$

by assumption on the choice of $\tilde{\sigma}_n$. This proves the weak version (without the factor $\sum_{l|k} \frac{\mu(l)}{l^{n-1}}$) of Theorem 3.3 by inequation (1) since $\sharp\{I \cap \mathbf{N}\} \longrightarrow \infty$ if $\mu \to \infty$.

We consider now intersections among the sets $I_1, I_2, \dots, I_A$ in order to deal with the factor $\sum_{l|k} \frac{\mu(l)}{l^{n-1}}$. This leads to a slightly better estimation of $\sharp(\bigcup_{k=1}^A I_k)$ and completes the proof of Theorem 3.3.

Call an element $x = (x_0, \dots, x_{n-1}) \in X_k(0)$ primitive if it is not of the form $h\mathbf{Z}^n$ for an integral divisor $h > 1$ of k. Call x imprimitive otherwise. An imprimitive element is of the form $h\tilde{x}$ with $\tilde{x} \in X_{k/h}(0)$ and contributes a common integer to the sets I_k and $I_{k/h}$. This implies the inequality

$$\sharp(\bigcup_{k=1}^A I_k) \leq \sum_{k=1}^A \sharp(X_k(0)_p)$$

where $X_k(0)_p \subset X_k(0)$ denotes the set of all primitive elements in $X_k(0)$.

It is thus enough to estimate the number of primitive elements in $X_k(0)$. We have

$$\sharp(X_k(*) \cap h\mathbf{Z}^n) \sim_{\mu\to\infty} \frac{1}{h^n}\sharp(X_k(*)).$$

We have obviously $X_k(a) \cap h\mathbf{Z}^n = \emptyset$ for $a \notin h\mathbf{Z}$. Applying Lemma 5.1, obviously modified, to the sublattice $h\mathbf{Z}^n \subset \mathbf{Z}^n$ of index h^n shows

$$\sharp(X_k(\alpha h) \cap h\mathbf{Z}^n) \sim_{\mu\to\infty} \frac{1}{k/h}\sharp(X_k(*) \cap h\mathbf{Z}^n)$$

for $\alpha = 0, 1, \dots, \frac{k}{h} - 1$. We get thus

$$\sharp(X_k(0) \cap h\mathbf{Z}^n) \sim_{\mu\to\infty} \frac{1}{kh^{n-1}}\sharp(X_k(*)) \sim_{\mu\to\infty} \frac{1}{h^{n-1}}\sharp(X_k(0)).$$

Since an element $x \in X_k(0) \cap h\mathbf{Z}^n$ belongs also to $X_k(0) \cap l\mathbf{Z}^n$ for any natural divisor l of h and since $\sum_{l|h} \mu(l) = 0$ for $h \geq 2$, the number $\sharp(X_k(0)_p)$ of primitive elements in $X_k(0)$ is asymptotically given by

$$\left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right) \sharp(X_k(0)).$$

This leads to the majoration

$$\sharp(\bigcup_{k=0}^A I_k) \leq_{\mu\to\infty} \sum_{k=1}^A \left(\sum_{l|k} \frac{\mu(l)}{l^{n-1}}\right) \sharp(X_k(0))$$

and proves Theorem 3.3. $\square$

Proof of Lemma 5.1 The statement of Lemma 5.1 is equivalent to the asymptotic equalities

$$\frac{\sharp(X_k(j)) - \sharp(X_k(i))}{\sharp(X_k(*))} \sim_{\mu\to\infty} 0$$

for $0 \leq i, j < k$.

Fix $0 \leq i < j < k$. Associate to an element $(x_0, x_1, \dots, x_{n-1}) \in X_k(j)$ the element $(x_0 + i - j, x_1, \dots, x_{n-1})$ provided that it belongs to $X_k(i)$. This induces a bijection between subsets $\tilde{X}_k(j)$ and $\tilde{X}_k(i)$ of $X_k(j), X_k(i)$. The set of "bad" points

$$B_k(i,j) = \left(X_k(j) \setminus \tilde{X}_k(j)\right) \cup \left(X_k(i) \setminus \tilde{X}_k(i)\right)$$

consists of some integral points at bounded Euclidean distance $< k \leq A$ from the boundary ∂Z_k of the the set

$$Z_k = \{(z_0, \dots, z_{n-1}) \in \mathbf{R}^n \mid \sum_{i=0}^{n-1} s_i z_i \in kI,\ \sum_{i=0}^{n-1} z_i^2 \leq \mu - k^2\}.$$

This shows that

$$|\sharp(X_k(j)) - \sharp(X_k(i))| \leq \sharp(B_k(i,j)) \leq \mathrm{vol}\left(N_{k+\sqrt{n}/2}(\partial Z_k)\right) \sim O(\mu^{n-1})$$

where $N_{k+\sqrt{n}/2}(\partial Z_k) \subset \mathbf{R}^n$ denotes the set of all points at distance $\leq k + \sqrt{n}/2$ from the boundary ∂Z_k of Z_k.

Since $\sharp(X_k(*)) = O(\mu^n)$ this proves Lemma 5.1. $\Box$

6 Proof of Theorem 1.5

Although an intuitively correct proof using Theorem 1.1 of Theorem 1.5 is easy, a rigorous proof is somewhat tedious and is the content of this section.

6.1 Two auxiliary results

The main ingredient for proving Theorem 1.5 is the following result which says essentially that an attracting fixpoint of a dynamical system is structurally stable.

Proposition 6.1 *Given a real interval $A \subset \mathbf{R}$, let $f_1, f_2, \dots : A \longrightarrow A$ be a sequence of uniformly converging functions with continuous and differentiable limit $f(x) = lim_{n\to\infty} f_n(x)$ on A. Suppose that f has a fixpoint $\xi = f(\xi) \in A$ and suppose that we have $sup_{x\in A}|f'(x)| = \lambda < 1$.*

Then the sequence $s_n(x)$ of functions defined recursively by $s_0(x) = x$ and $s_n(x) = f_n(s_{n-1}(x))$ converges pointwise to the constant function ξ.

Proof Given $\delta > 0$ there exists an integer N such that $|f_n(x) - f(x)| < \delta(1-\lambda)$ for all $x \in A$ and for all $n > N$. We have then for $m > N$

$$\begin{array}{rcl} |s_m(x) - \xi| & = & |f_m(s_{m-1}(x)) - \xi| \\ & < & \delta(1-\lambda) + |f(s_{m-1}(x)) - \xi| \\ & < & \delta - \lambda\delta + \lambda|s_{m-1}(x) - \xi| \\ & < & \delta - \lambda^2\delta + \lambda^2|s_{m-2}(x) - \xi| \\ & & \vdots \\ & < & \delta - \lambda^{m-N}\delta + \lambda^{m-N}|s_N(x) - \xi| \ . \end{array}$$

This shows $|s_m(x) - \xi| < 2\delta$ if

$$m > \max(N, N + \log\left(\frac{\delta}{|s_N(x) - \xi|}\right)/\log(\lambda))$$

and implies the result since we can choose $\delta > 0$ arbitrarily small. $\square$

Remark 6.2 *(i) The proof of Proposition 6.1 shows in fact*

$$|s_n(x) - \xi| \leq \lambda^n|x - \xi| + \sum_{k=1}^{n} \lambda^{n-k} sup_{x\in A}|f_k(x) - f(x)| \ .$$

Asymptotically, we have thus $|s_n(\xi) - \xi| = O(sup_{x\in N(\xi)}|f_n(x) - f(x)|)$ *(where* $N(\xi) \subset A$ *is an arbitrarily small fixed neighbourhood of* ξ*) if* $sup_{x\in N(\xi)}|f_n(x) - f(x)|$ *is decreasing at a slower rate than powers of* λ*.*

(ii) If the sequence $f_n(x) = F(x, 1/n)$ *satisfies the hypotheses of Proposition 6.1 with* $F(x, y)$ *having continuous partial derivatives of all orders up to* $k+1$ *in a neighbourhood of* $(\xi, 0)$*, then there exist constants* $a_1, a_2, \ldots, a_k$ *such that*

$$s_n(x) = \xi + \frac{1}{1 - \partial/\partial_x F}\left(\sum_{j=1}^{k} \frac{a_j}{j!} n^{-j}\right) + O(n^{-(k+1)})$$

where $\frac{\partial^{a+b} F}{\partial x^a \partial y^b}$ *denotes the obvious partial derivative of* $F(x, y)$*, evaluated at* $(\xi, 0)$*. The formulae for the first three coefficients* a_1, a_2, a_3 *are*

$$\begin{array}{rcl} a_1 & = & \partial/\partial_y F \\ a_2 & = & 2a_1 + (a_1\partial/\partial_x + \partial/\partial_y)^2 F \\ a_3 & = & 12a_2 - 6a_1 + 6\partial/\partial_y (a_1\partial/\partial_x + \partial/\partial_y) F \\ & & \quad + (a_1\partial/\partial_x + \partial/\partial_y)^3 F \end{array}$$

In particular, for $F(x, y)$ *analytic and non-constant in* y*, the sequence* $s_n(x)$ *is asymptotically independent from* x*.*

For $x \in (0, \infty)$ we consider the real analytic positive function

$$\tau(x) = \sum_{k=1}^{\infty} e^{-\pi(k/x)^2} = \frac{1}{2}\theta_3(\frac{i}{x^2}) - \frac{1}{2} ,$$

related to the third Jacobi-theta function $\theta_3(z) = \sum_{k \in \mathbf{Z}} e^{i\pi k^2 z}$, cf. for instance Equation (6), page 102 in [5]. For $x > 0$, we have $\tau'(x) = \frac{2\pi}{x^3} \sum_{k=1}^{\infty} k^2 e^{-\pi(k/x)^2} > 0$ and the easy inequalities

$$\frac{x}{2} - 1 < \int_0^\infty e^{-\frac{\pi}{x^2}t^2} dt - \int_0^1 e^{-\frac{\pi}{x^2}t^2} dt < \sum_{k=1}^{\infty} e^{-\pi(k/x)^2} < \int_0^\infty e^{-\frac{\pi}{x^2}t^2} dt = \frac{x}{2}$$

for $x > 0$ imply that $x \longmapsto \tau(x)$ is an increasing analytic diffeomorphism of $(0, \infty)$. The equation

$$\frac{1}{x} = \tau\left(\frac{\Omega(x)}{x}\right) = \sum_{k=1}^{\infty} e^{-k^2\pi(x/\Omega(x))^2}$$

defines thus a real positive analytic function $\Omega : (0, \infty) \longrightarrow \mathbf{R}$. Equivalently, the function Ω is given by $\Omega(x) = x\psi\left(\frac{1}{x}\right)$ where the analytic diffeomorphism ψ satisfies $\psi(\tau(x)) = \tau(\psi(x)) = x$ for all $x > 0$ and is the reciprocal function of τ.

The proof of Theorem 1.5 uses the following result.

Proposition 6.3 *The application*

$$x \longmapsto \Omega(x) = x\psi\left(\frac{1}{x}\right)$$

defines a continuous map from $(0, \infty)$ onto $(2, \infty)$ which is strictly increasing for $x > 2$. It has a unique fixpoint $\xi = \frac{1}{\tau(1)} = \frac{1}{\sum_{k=0}^{\infty} e^{-\pi k^2}} \sim 23.13882534$ which is attracting under iteration since

$$\Omega'(\xi) = 1 - \frac{\tau(1)}{\tau'(1)} = 1 - \frac{\sum_{k=1}^{\infty} e^{-\pi k^2}}{2\pi \sum_{k=1}^{\infty} k^2 e^{-\pi k^2}} \sim 0.9135652 < 1 .$$

6.2 Proof of Theorem 1.5

Given an $(n-1)-$dimensional lattice of density $\tilde{\Delta}_{n-1}$, Theorem 3.3 implies the existence of an $n-$dimensional lattice with density $\tilde{\Delta}_n$ arbitrarily close to $\frac{1}{2^n \overline{\sigma}}$ for $\overline{\sigma} > 0$ defined by

$$2^{n-1}\tilde{\Delta}_{n-1} \sum_{k=1}^{A} \sqrt{1 - k^2 \left(2^{n-1}\tilde{\Delta}_{n-1}\frac{V_n}{V_{n-1}}\overline{\sigma}\right)^2}^{\,n-1} = 1$$

where

$$A = \left\lfloor \frac{2^{1-n} V_{n-1}}{V_n \tilde{\Delta}_{n-1} \overline{\sigma}} \right\rfloor \ .$$

Given a positive constant $\epsilon > 0$ and a natural integer $N \in \mathbf{N}$, there exists thus a sequence of lattices $\Lambda_1, \Lambda_2, \dots, \Lambda_N$ of dimensions $1, 2, \dots, N$ with densities $\tilde{\Delta}_1 = 1, \tilde{\Delta}_2, \dots, \tilde{\Delta}_N$ satisfying

$$\tilde{\Delta}_m \geq (1-\epsilon) \frac{d_m}{2^m}, \ m = 1, \dots, N$$

where $d_1 = 2$ and $d_2, d_3, \dots, d_N$ are recursively defined by the equation

$$d_{n-1} \sum_{k=1}^{A_n} \sqrt{1 - k^2 \left(\frac{d_{n-1}}{d_n} \frac{V_n}{V_{n-1}} \right)^2}^{\,n-1} = 1 \text{ with } A_n = \left\lfloor \frac{d_n \ V_{n-1}}{d_{n-1} \ V_n} \right\rfloor \ .$$

Equivalently, the sequence $d_1, d_2, \dots$ is given by $d_1 = 2, d_2 = f_1(2), d_3 = f_2(d_2), \dots, d_{n+1} = f_n(d_n), \dots$ where $f_1, f_2, \dots : (0, \infty) \longrightarrow (0, \infty)$ are the functions defined implicitely by the equations

$$x \sum_{k=0}^{\lfloor f_n(x) V_n / (x V_{n+1}) \rfloor} \sqrt{1 - k^2 \left(\frac{x V_{n+1}}{f_n(x) V_n} \right)^2}^{\,n} = 1 \ .$$

Stirlings formula $n! = \sqrt{2\pi n}(n/e)^n(1 + O(1/n))$ shows

$$V_{n+1}/V_n = \sqrt{\pi} \frac{(n/2)!}{((n+1)/2)!} = \sqrt{2\pi/n} \ (1 + O(1/n)) \ .$$

We have thus asymptotically

$$\begin{aligned} 1 = & \ x \sum_{k=0}^{\lfloor f_n(x) V_n / (x V_{n-1}) \rfloor} \sqrt{1 - k^2 \left(\frac{x V_{n+1}}{f_n(x) V_n} \right)^2}^{\,n} \\ = & \left(x \sum_{k=1}^{\infty} e^{-k^2 \pi (x / f_n(x))^2} \right) (1 + O(1/n)) \end{aligned}$$

and $f_n(x) \longrightarrow \Omega(x)$ uniformly on any compact subset of $(0, \infty)$. By Proposition 6.3 we can find $\alpha < \xi = \left(\sum_{k=1}^{\infty} e^{-\pi k^2} \right)^{-1} \sim 23.14 < \beta$ such that $\Omega'(x) \leq 19/20$ for $x \in [\alpha, \beta]$. We have thus uniform convergency $f_n(x) \longrightarrow \Omega(x)$ for $x \in [\alpha, \beta]$, and there exists an integer N_ξ such that $f_n([\alpha, \beta]) \subset [\alpha, \beta]$ for all $n \geq N_\xi$. Proposition 6.1 shows now

$$\lim_{n \to \infty} d_n = \xi$$

which ends the proof. □

The following Table illustrates the convergence of the sequence $d_1 = 2, d_2 = f_1(d_1), \dots$:

1	2.00000000	2.00000000	0
2	3.62759873	3.99997210	−0.7447467
4	8.08369319	7.92472241	0.6358831
8	18.71971890	14.38756801	34.6572071
16	30.69030131	20.71395996	159.6214617
32	29.45114255	22.98242063	206.9991014
64	25.53248635	23.13821340	153.2334688
128	24.17810739	23.13882533	133.0281029
256	23.63011883	23.13882534	125.7711333
512	23.37820694	23.13882534	122.5633803
1024	23.25703467	23.13882534	121.0463495

The first column shows the indices n, choosen as successive powers of 2. The second column shows the corresponding value of d_n. The third column shows the $(n-1)-$th iteration of Ω, starting from the initial value 2. The last column is the difference between the second and third column, multiplied by n and illustrates the expected finer asymptotic properties.

Asymptotically, the number d_n is roughly given by

$$23.13882534 + 119.58193\frac{1}{n} + 1473.8282\frac{1}{n^2} + 25774.448\frac{1}{n^3} + \dots$$

(cf assertion (ii) of Remark 6.2).

6.3 Proof of Proposition 6.3

Using the orientation-reversing diffeomorphism $x = \frac{1}{\tau(Y)} \longmapsto Y = \psi\left(\frac{1}{x}\right)$ of $(0, \infty)$ we have

$$\frac{Y}{\tau(Y)} = \Omega(x) = \Omega\left(\frac{1}{\tau(Y)}\right) .$$

The inequality $\tau(Y) < \frac{Y}{2}$ shows $\Omega(x) = \frac{Y}{\tau(Y)} > 2$ and $\frac{2Y}{Y-2} > \frac{Y}{\tau(Y)}$ implies $\lim_{x\to 0_+}\Omega(x) = 2$. Since

$$\lim_{Y\to 0_+}\frac{Y}{\tau(Y)} = \lim_{Y\to 0_+} Y e^{\pi^2/Y^2}\left(1 + \sum_{k=2}^{\infty} e^{-\pi(k^2-1)/Y^2}\right)^{-1} = \infty$$

the map Ω is a surjection onto $(2, \infty)$.

Since $x \longmapsto Y$ is orientation reversing, $\frac{d}{dx}\Omega(x) > 0$ is equivalent to strict positivity of

$$Y^2\frac{d}{dY}\left(\frac{\tau(Y)}{Y}\right) = Y\tau'(Y) - \tau(Y) = \frac{1}{Y^2}\sum_{k=1}^{\infty}\left(2\pi k^2 - Y^2\right)e^{-\pi(k/Y)^2}$$

which obviously holds for $Y \leq \sqrt{2\pi}$ corresponding to

$$x \geq \frac{1}{\tau(\sqrt{2\pi})} = \frac{1}{\sum_{k=1}^{\infty} e^{-\pi k^2/2}} \sim 1.38 .$$

This implies that Ω restricts to an increasing diffeomorphism from $(2, \infty)$ onto $(\Omega(2), \infty)$ and since $\Omega(x) > 2$, the map $x \longmapsto \Omega(x)$ has a unique fixpoint at $\xi = \frac{1}{\tau(1)}$. $\Box$

7 Final remarks

The inequality

$$\sharp(\bigcup_{k=1}^{A} I_k) \leq \sum_{k=1}^{A} \sharp(X_k(0)_p)$$

appearing in the proof of Theorem 3.3 is probably not sharp. A smaller upper bound for the cardinality $\sharp(\bigcup_{k=1}^{A} I_k)$ would thus improve the results of this paper.

The inequality above can be decomposed into the two inequalities

$$\sharp(\bigcup_{k=1}^{A} I_k) \leq \sum_{k=1}^{A} \sharp(I_{k,p})$$

and

$$\sharp(I_{k,p}) \leq \sharp(X_k(0)_p)$$

where we denote by $I_{k,p} \subset I_k$ the subset of integers corresponding to primitive elements. If the subsets $I_{1,p}, \dots, I_{A,p}$ are asymptotically "independent" in the sense that

$$\sharp(\bigcap_{j=1}^{l} I_{k_j,p})/\sharp(I \cap \mathbf{Z}) \sim_{\mu \to \infty} \prod_{j=1}^{l} \left(\sharp(I_{k_j,p})/\sharp(I \cap \mathbf{Z})\right) ,$$

for $\{I_{k_1,p}, \dots, I_{k_l,p}\} \subset \{I_{1,p}, \dots, I_{A,p}\}$ a subset of l distinct elements, one can neglect the contributions corresponding to $k = 2, \dots, A$. This would lead to a small improvement.

A probably much more important improvement would result from a better understanding of the inequality $\sharp(I_{k,p}) \leq \sharp(X_k(0)_p)$.

Instead of working with sublattices of $\mathbf{Z}^{n+1}$ orthogonal to a given vector $(s_0, \dots, s_n) \in \mathbf{Z}^{n+1}$, it is possible to consider sublattices $\mathbf{Z}^{n+a}$ which are orthogonal to a set of $a \geq 2$ linearly independent vectors in $\mathbf{Z}^{n+a}$. One might also replace the standard lattice $\mathbf{Z}^{n+1}$ by other lattices, e.g. sublattices of dimension n in $\mathbf{Z}^{n+1}$ (which approximate homothetically an arbitrary lattice by Proposition 2.2) or of finite index in $\mathbf{Z}^{n+1}$.

Extending finite $\mu-$sequences in an optimal way into longer $\mu-$sequences amounts geometrically to the familiar process of lamination for lattices (see for instance [5] or [9]). The existence of an integer $s \in I \setminus I_1$ implies indeed the existence of a point $P \in \mathbf{E}^{n-1}$ which is far away from any lattice point of the affine lattice $\{(x_0, \dots, x_{n-1}) \mid \sum x_i s_i = s\} \subset \mathbf{Z}^n$ and corresponds thus to a "hole" of the lattice.

The present version of this paper ows much to Fedor Petrov whose pertinent questions clarified and improved (and changed the title of) a preliminary version, see [1].

I thank also A. Marin, J. Martinet, P. Sarnak, B. Venkov and J-L. Verger-Gaugry for helpful comments and interest in this work.

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